E2 - 95 - 503

MULTI-INSTANTON EFFECTS IN QCD SUM RULES FOR THE PION*

A. E. Dorokhov, S. V. Esaibegyan†, N. I. Kochelev, and N. G. Stefanis‡

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,

141980 Dubna, Moscow Region, Russia

(February 1, 2008)

Abstract

Multi-instanton contributions to QCD sum rules for the pion are investigated within a framework which models the QCD vacuum as an instanton liquid. It is shown that in singular gauge the sum of planar diagrams in leading order of the $1/N_c$ expansion provides similar results as the effective single-instanton contribution. These effects are also analysed in regular gauge. Our findings confirm that at large distances the correlator functions are more adequately described in the singular gauge rather than in the regular one.

*Work supported in part by the Heisenberg-Landau program

†On leave of absence from Yerevan Physics Institute, Yerevan, Armenia

‡On leave of absence from Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany

The QCD sum rule approach [1] allows the investigation of hadron properties in a systematic manner. It provides, in particular, the possibility to describe static characteristics of particles, such as masses, decay constants, form factors, etc., in an energy region where perturbative methods are not applicable [2–6]. Within this approach, the effects of large distances are effectively parametrized in terms of local matrix elements of quark-gluon operators averaged over the physical vacuum (vacuum condensates), quantities which are independent of hadron properties. On the other hand, short-distance physics is contained in the Wilson coefficients of the Operator Product Expansion (OPE) entering the calculation of correlators.

However, the nonperturbative matrix elements contain contributions that are not taken into account in the OPE. These are, so-called, "direct" small-size instantons (see, e.g., [7]) which give essential nonlocal contributions to current correlators in the channels where they are allowed by quantum numbers. These contributions are not sufficiently accounted for in the *local* condensates, since the latter correspond to vacuum fluctuations with infinite correlation length. They should rather be taken into account within the Wilson coefficients along with the factors calculated by perturbative methods.

The instanton liquid model of the QCD vacuum, originally suggested in [8,9], has later been further generalized by an analytic approach, based on the Feynman variational principle [10–12]. (For lattice calculations using this vacuum model, see, e.g., [13].)

As it was shown in [12], the instanton-induced vacuum fluctuations are responsible for the spontaneous breaking of chiral invariance. This chiral-symmetry-breaking mechanism is based on the idea of mixing and delocalization of fermion zero modes in the field of the instanton (I) and anti-instanton (A) pairs. The QCD vacuum is modeled as an $I-A$ diluted liquid, characterized by a small ratio $\rho_c/R \simeq 1/3$, where $\rho_c \simeq 1/600$ MeV$\simeq 1/3$ Fm is the average instanton size in the vacuum, and R is the average distance between pseudoparticles.

A summary of some successful applications of this approach includes: The calculation of current correlators in the background of I and A external fields which provides a useful procedure for extracting the static features of the pseudoscalar meson octet [8,14]. More re-

cently [15], a possible mechanism for the bound-state formation in the vector-meson channel has been proposed. In a series of works [16], several main properties of hadron spectroscopy have been quantitatively determined. Evidence was provided there that large spin-flip high-energy amplitudes [17] are the result of the spin-dependent interaction between quarks, induced by the small-size vacuum fluctuations.

The role of direct instantons in stabilizing the QCD sum rules for the nucleon [2–4] was first discussed in [18] and later also in [19]. These analyses show that the inclusion of the instanton contributions amount to a significant enlargement of the stability region of the Borel parameter.

The instanton contribution to different vacuum matrix elements is defined basically by the quark zero modes in external I , A fields. Due to the specific chiral and flavor properties of these fields, instanton effects depend strongly on the channel under consideration. In the channel with the quantum number 0^-, the instanton contribution is dominant [7]. The single instanton contribution to the QCD sum rule for the pion, within the effective approach given in [9,20], has been first calculated in [9]. He has shown that a self-consistent description of the pion as a pseudo-Goldstone mode is possible only if the contribution of direct instantons is taken into account.

It is the purpose of this paper to investigate the multi-instanton contributions to QCD sum rules for the pion in the framework proposed in [12]. The main conclusion of this investigation is that the large-distance behavior of the pion correlator in the singular gauge is essentially the same as in the effective single instanton approach [9]. The behavior of the correlator in the regular gauge is also explored but found to give a negligible contribution at large distances.

The QCD sum rules for the pion are evaluated from the correlator function

$$\Pi(q) = i \int d^4x \, \mathrm{e}^{iqx} \langle 0|T(j(x)j^+(0)|0\rangle \ , \tag{1}$$

which is considered at $Q^2 = -q^2 \simeq 1$ GeV. We will analyze the sum rules for a charged pion, so that

$$j(x) = q_u q_{\bar{d}} \left[\bar{v}_R i\gamma_5 u_L + \bar{v}_L i\gamma_5 u_R \right](x) \ . \tag{2}$$

Here q_i denotes quark annihilation operators, and $u_{L(R)} = \left(\frac{1\pm\gamma_5}{2}\right)$ are left- (right-) handed spinors.

The single instanton contribution has been computed in [9], assuming that the quark Green function in the background of the instanton field

$$S_I(x,y) = S_0(x,y) + S_{\pm}(x,y) \tag{3}$$

can be approximated by the expression

$$S_{\pm}(x,y) = \langle q^a_\alpha(x) \bar{q}^b_\beta(y) \rangle = \int d^4z \, \frac{\left[\Psi^{\pm}_z(x) \bar{\Psi}^{\pm}_z(y)\right]^{ab}_{\alpha\beta}}{m^*} \tag{4}$$

which retains only the zero modes, given in singular gauge by

$$\Psi^{\pm}_{x_0}(x) = \Phi(x - x_0) \frac{1 \pm \gamma_5}{2} (\not{x} - \not{x}_0) U \tag{5}$$

with $\Phi(x) = \frac{\rho_c}{\sqrt{x^2}\pi[x^2+\rho_c^2]^{3/2}}$, and where we have averaged over instanton positions, denoted by x_0. In Eq. (4), (a,b) are color and (α,β) spinor indices, respectively; U is the color-spin matrix ($U^+U = 1$), whereas $\pm$ refers to the instanton (anti-instanton). The effective quark mass m^*, acquired in the instanton vacuum [9,20], is

$$m^* \approx -\frac{2}{3}\pi^2 \langle 0|\bar{q}q|0\rangle \rho_c^2 \simeq 200 \text{ MeV} \ . \tag{6}$$

Note that the free quark propagator in (3)

$$S_0(x,y) = \frac{i(\not{x} - \not{y})}{2\pi^2 (x-y)^4} \tag{7}$$

serves to approximately account for the contribution of the non-zero modes.

Applying now the Borel transformation [1,9]

$$B[f(s)] \equiv \lim_{\substack{n\to\infty \\ s\to\infty}\, n/s=\tau^2} (-1)^n \frac{s^{n+1}}{n!} \left(\frac{d}{ds}\right)^n f(s) \ , \tag{8}$$

and using the expression for the instanton density [9,11], $n_c(\rho)$,

$$
\begin{aligned}
n(\rho) &= n_c \delta(\rho - \rho_c) \\
n_c &\simeq 0.8 \times 10^{-3}\ \mathrm{GeV}^4\ ,
\end{aligned}
\tag{9}
$$

derived in the instanton liquid model, in conjunction with the relation $\langle \bar{q} q \rangle = -\frac{2n_c}{m^*}$ between the instanton density and the quark condensate, we obtain the following correlator in terms of the inverse Borel parameter $\tau^2 = 1/M^2$

$$
\Pi(\tau) = \frac{2n_c\, \rho_c^2\, \zeta}{m^{*2}\, \tau^4 \sqrt{\pi}} \int_0^\infty d\alpha \int_0^\infty d\beta\, \mathrm{e}^{-\zeta^2 t^2} \left(\zeta^2 t^3 - \frac{3}{2} t \right) \cosh\alpha\ \cosh\beta\ , \tag{10}
$$

where $t = \frac{\cosh\alpha + \cosh\beta}{2}$, and $\zeta = \rho_c \tau$. This result differs from the one given in [9] by a factor $\sqrt{\frac{\pi}{2}}$ (which may be a misprint there). Employing the substitutions $\frac{\alpha+\beta}{2} = y_1$, $\frac{\alpha-\beta}{2} = y_2$ in Eq. (10), the double integral can be further expressed via the MacDonald functions $K_\nu(\frac{\zeta^2}{2})$ to read

$$
\Pi(\tau) = \frac{3}{8} \frac{\zeta^2}{\tau^4\, \pi^2}\, \mathrm{e}^{-\zeta^2/2} \left[K_0 \left(\frac{\zeta^2}{2} \right) + K_1 \left(\frac{\zeta^2}{2} \right) \right]\ , \tag{11}
$$

It is worth remarking again that, as it was shown in [9], the QCD sum rule in the pseudoscalar channel can be saturated only by including the one-instanton contribution (cf. Eq. (10)).

As it was in [21], the correction to $\Pi(\tau)$, arising from the next to leading $I{-}A$ contribution is at the level of 5% relative to the leading contribution given by Eq. (11) when τ is large, i.e., $\tau \simeq \rho_c$. This τ-region of the correlator corresponds to large distances and hence to exploit the QCD sum rules, one may use the techniques developed in [10–12], based as already said on the summation of planar diagrams in leading $1/N_c$ approximation. Suffice to say that such calculations are based on a model Green function for the quark in the background field of one instanton which actually resembles Eq. (4) with the effective quark mass m^* now being replaced by the current quark mass m. The summation based on the delocalization mechanism of zero modes [11,12] gives an expression for the two-point function $\Pi(q)$ (cf. Eq. (1)). At small momentum q, the connected part of this function defines the pion mass and the value of the decay constant f_π, in good agreement with the data (see also [14,22,23]).

In the pseudoscalar channel, the expression for $\Pi(q)$ which incorporates the multi-instanton/anti-instanton effects is

$$\Pi(q) = \frac{4\,V\,N_c^2}{N}\,\Gamma_5^2(q)\,\frac{1}{R_-(q)} \tag{12}$$

indicating the pole position in the γ_5-channel. Here, N is the number of instantons, N_c the number of colors, V the space-time volume, and $\Gamma_5^2(q)$ an effective vertex function. In the analysis of [12], the behavior of $\Pi(q)$, expressed in Eq. (12), has been investigated in the limit $q \longrightarrow 0$. In the present work, we are primarily interested in the limit $q^2 \simeq 1$ GeV^2, where the QCD sum rules can be safely evaluated [1–5]. However, in this region of momentum the estimates derived from Eq. (4) with the replacement $m^* \rightarrow m$ are valid at a qualitatively level only [12]. Recall that the model Green function "works" well in the two limiting cases $\rho q \gg 1$ and $\rho q \ll 1$.

Then in leading $\rho_c M(0)$ approximation we have

$$\begin{aligned}
R_-(q) \quad &= \quad \frac{2\,V\,N_c}{N}\int\frac{d^4k}{(2\pi)^4}\,\frac{M_1^2k_2^2 + M_2^2k_1^2 - 2k_1k_2M_1M_2}{(M_1^2+k_1^2)(M_2^2+k_2^2)} \\
&\underset{Q^2\simeq 1\ \mathrm{Gev}^2}{=} \frac{4\,V\,N_c}{N}\int\frac{d^4k}{(2\pi)^4}\,\frac{M^2(k)}{k^2+M^2(k)} \\
&\simeq \quad \frac{4\,V\,N_c\;M^2(0)}{N}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+M^2(0)} \\
&\simeq \quad \frac{4\,V\,N_c\,M^2(0)\pi^2}{(2\pi)^4\,N\,\rho_c^2}
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
\Gamma_5(q) \quad &= \quad -\int\frac{d^4k}{(2\pi)^4}\,\frac{\sqrt{M(k)M(k+q)}\,(k^2+(k+q)^2-q^2)}{[M^2(k)+k^2][M^2(k+q)+(k+q)^2]} \\
&\underset{Q^2\simeq 1\ \mathrm{GeV}^2}{\simeq} - \frac{\sqrt{M(0)M(Q)}\pi^2}{(2\pi)^4\,2Q^2\,\rho_c^4}\;,
\end{aligned} \tag{14}$$

where $M_{1(2)}$ is a short-hand notation for $M(k \mp \frac{q}{2})$. Assuming fixed values of the instanton radii, $\rho = \rho_c$, it follows $M(p) \sim p^2\varphi^2(p)$, $\varphi(p)$, being associated with the zero mode representation in momentum space. It has the following asymptotics

$$\varphi_s(p) = \begin{cases} -\frac{2\pi\rho}{|p|}, & \rho p \ll 1 \\ -\frac{12\pi}{p^4\rho^2}, & \rho p \gg 1 \end{cases} \tag{15}$$

$$\varphi_r(p) = \begin{cases} \frac{4\pi\rho}{|p|}, & \rho p \ll 1 \\ -\frac{4\pi\rho}{p}\,\mathrm{e}^{-p\rho}, & \rho p \gg 1 \end{cases} \tag{16}$$

in singular and regular gauges, respectively. Since $M(p)$ is a rapidly increasing function with p for $\rho p \gg 1$ [11,24], we cut off the k-integration in (13) and (14) at values $\sim 1/\rho^2$.

Then by using Eq. (4) in conjunction with Eqs. (12)-(14), we obtain

$$\Pi(q) = \frac{N_c}{Q^4}\,\frac{M(Q)}{M(0)}\,\frac{1}{2^6\rho^6\pi^2}\ , \tag{17}$$

and utilizing the explicit expressions for $\varphi(p)$, given in [11,23], viz.

$$\begin{aligned} \varphi_s(p) &= \pi\,\rho^2 \frac{d}{d\xi}\left[I_0(\xi)K_0(\xi) - I_1(\xi)K_1(\xi)\right]_{\xi=\frac{\rho p}{2}}\ , \\ \varphi_r(p) &= \frac{4\pi\rho}{p}\ \mathrm{e}^{-p\rho}\ , \end{aligned} \tag{18}$$

we find the corresponding results for the correlators

$$\begin{aligned} \Pi(q)^{sing} &= \frac{N_c}{Q^2\,\rho^4\,2^6\pi^2}\left[I_1(\xi)K_0(\xi) - I_0(\xi)K_1(\xi) + \frac{I_1(\xi)K_1(\xi)}{\xi}\right]^2_{\xi=\frac{Q\rho}{2}} \\ \Pi(q)^{reg} &= \frac{N_c}{Q^4\,\rho^6\,2^6\pi^2}\ \mathrm{e}^{-2Q\rho}\ . \end{aligned} \tag{19}$$

Using now the integral representations

$$K_\nu(\xi) = \frac{(\frac{\xi}{2})^\nu\,\Gamma(1/2)}{\Gamma(\nu+1/2)}\int_1^\infty \mathrm{e}^{-\xi t}\,(t^2-1)^{\nu-1/2}dt\ , \tag{20}$$

$$I_\nu(\xi) = \frac{(\frac{\xi}{2})^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_0^\pi \mathrm{e}^{\pm\xi\cos\theta}\ \sin^{2\nu}(\theta)\ d\theta\ , \tag{21}$$

and the following Borel transforms, in accordance with Eq. (8),

$$\begin{aligned} B[\mathrm{e}^{-a\sqrt{s}}] &= \frac{a}{\sqrt{4\pi\tau^3}}\ \mathrm{e}^{-\frac{a^2}{4\tau^2}} \\ B\left[\frac{1}{s^2}\,\mathrm{e}^{-a\sqrt{s}}\right] &= \left(\tau^2 + \frac{a^2}{2}\right)\left[1 - \mathrm{erf}\left(\frac{a}{2\tau}\right)\right] - \frac{\tau a}{\sqrt{\pi}}\,\mathrm{e}^{-\frac{a^2}{4\tau^2}}\ , \end{aligned} \tag{22}$$

it follows

$$\Pi^{sing}(\tau) = \frac{N_c}{8(2\pi)^4\sqrt{\pi}\zeta\tau^4}\ I\ , \tag{23}$$

where

$$I = \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_1^\infty dt_1 \int_1^\infty dt_2 \, C \, t \, \mathrm{e}^{-\frac{\zeta^2 t^2}{16}}; \tag{24}$$

with $t = \cos\theta_1 + \cos\theta_2 + t_1 + t_2$ and

$$\begin{aligned} C = & \left(\frac{1}{8} - \frac{1}{4}\sin^2\theta_1 + 2\sin^2\theta_1 \sin^2\theta_2\right)\sqrt{t_1^2-1}\sqrt{t_2^2-1} \\ & -\frac{1}{4}\sin^2\theta_1\cos^2\theta_2 \frac{\sqrt{t_1^2-1}}{\sqrt{t_2^2-1}} + \frac{1}{8}\sin^2\theta_1\sin^2\theta_2 \frac{1}{\sqrt{t_1^2-1}\sqrt{t_1^2-1}} \ . \end{aligned} \tag{25}$$

The values of the integral I are tabulated below.

ζ	1	1.5	2	2.5	3
I	279	26.4	4.4	1	0.29

The analogous result to Eq. (23) in regular gauge is

$$\Pi^{reg}(\tau) = \frac{N_c}{16(2\pi)^2\zeta^6\tau^4} f(\zeta) \tag{26}$$

with

$$f(\zeta) = \left(1 + 2\zeta^2\right)[1 - \mathrm{erf}(\zeta)] - \frac{2\zeta}{\sqrt{\pi}} \mathrm{e}^{-\zeta^2} \ , \tag{27}$$

where

$$\mathrm{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta \mathrm{e}^{-y^2} dy \ . \tag{28}$$

Comparing these results with the effective single-instanton contribution, given by Eq. (11), at $\tau = \rho$, we deduce

$$\Pi^{sing}_{mult.}(\tau) \approx \frac{7}{11} \Pi^{sing}_{eff.}(\tau) \ . \tag{29}$$

Taking into account that the accuracy of the results obtained by QCD sum rules is limited by uncertainties on the order of 30%, we may claim that within the context of assuming the validity of Eq. (4), both approaches give coincident results in singular gauges for the pion correlator $\Pi(\tau)$ at large τ. This means that in the region of large distances, multi-instanton

contributions to the correlator, obtained via summation over planar diagrams [11,12] on one hand, and those from the analysis [9,20], based on an effective single-instanton approach on the other hand, are actually two different languages which correctly describe the same phenomena. Perhaps even more importantly, both methods achieve saturation of the QCD sum rules only by including in the pseudoscalar channels the instanton contribution. This conforms with the assumption that the QCD vacuum is dominated by small-size instantons.

As regards the regular gauge, evaluation of $\Pi^{reg}(\tau)$ (cf. (26)) amounts to a very small value relative to $\Pi^{sing}(\tau)$ in the region where the QCD sum rules apply. A strong cancellation in regular gauge at large distances was also pointed out in [24]. The investigation presented in this work provides further arguments in favor of a singular gauge when processes at low energies (i.e., large distances) are studied.

ACKNOWLEDGMENTS

The work of S. V. E. was supported in part by Grant 93-283. One of us (N. G. S.) thanks the members of the Joint Institute for Nuclear Research for the warm hospitality extended to him during his stay.

REFERENCES

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979); **B147**, 519 (1979).

[2] B. L. Ioffe, Nucl. Phys. **B188**, 317 (1981); **B191**, 591(E) (1981).

[3] V. M. Belyaev and B. L. Ioffe, Zh. Eksp. Teor. Fiz. **83**, 876 (1982) [Sov. Phys. JETP **56**, 493 (1982)].

[4] Y. Chung *et al.*, Nucl. Phys. **B197**, 55 (1982); Z. Phys. C **15**, 367 (1982).

[5] B. L. Ioffe and A. V. Smilga, Nucl. Phys. **B216**, 373 (1983).

[6] V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. **115B**, 410 (1982).

[7] V. A. Novikov *et al.*, Sov. J. Part. Nucl. Phys. **13**, 542 (1982).

[8] E. V. Shuryak, Nucl. Phys. **B203**, 93 (1982); **B203**, 116 (1982); **B203**, 140 (1982).

[9] E. V. Shuryak, Nucl. Phys. **B214**, 237 (1983); Phys. Rep. **115**, 151 (1984).

[10] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B245**, 259 (1984).

[11] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B272**, 457 (1986).

[12] D. I. Diakonov and V. Yu. Petrov, Zh. Eksp. Teor. Fiz. **89**, 361 (1985); **89**, 751 (1985).

[13] M. C. Chu and S. Huang, Phys. Rev. D **45**, 2446 (1992).

[14] S. V. Esaibegyan and C. N. Tamaryan, Yad. Fiz. **49**, 815 (1989); **55**, 2193 (1992); **58**, 1 (1995).

[15] S. V. Esaibegyan and C. N. Tamaryan, Pis'ma v JETF **61**, 3 (1995).

[16] N. I. Kochelev, Yad. Fiz. **41**, 456 (1985) [Sov. J. Nucl. Phys. **41**, 291 (1985)]; A. E. Dorokhov and N. I. Kochelev, Yad. Fiz. **52**, 214 (1990) [Sov. J. Nucl. Phys. **52**, 135 (1990)]; A. E. Dorokhov, N. I. Kochelev, and Yu. Zubov, Sov. J. Part. Nucl. Phys. **23**,

522 (1992); A. E. Dorokhov, Nucl. Phys. **A581**, 654 (1995).

[17] A. E. Dorokhov, N. I. Kochelev, and Yu. Zubov. Int. J. Mod. Phys. **A5**, 603 (1993); and references cited therein.

[18] A. E. Dorokhov and N. I. Kochelev, Z. Phys. C **46**, 281 (1990).

[19] H. Forkel and M. K. Banerjee, Phys. Rev. Lett. **71**, 484 (1993).

[20] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B163**, 45 (1980).

[21] A. E. Dorokhov, S. V. Esaibegyan, and N. I. Kochelev, submitted to Yad. Fiz..

[22] E. V. Shuryak, Rev. Mod. Phys. **61**, 1 (1993).

[23] M. Hutter, München preprint # LMU-95-01 (1995).

[24] M. Hutter, München preprint # LMU-95-03 (1995).